Exploring Flat One-Vertex Folds

Overview

Participants create a variety of flat one-vertex folds and examine the crease patterns that appear when the papers are opened. Participants count mountain and valley folds and use a protractor to measure angles in their quest to create conjectures about the crease patterns.

Known patterns that participants may rediscover include the Maekawa-Justin Theorem (there are always two more mountains than valleys or vice versa), the Even Degree Theorem (the total number of mountains and valleys is always even), the Big-Little-Big Theorem (if there is a little angle between two larger angles then one of the separating creases must be a mountain and the other must be a valley), the Kawasaki-Justin Theorem (the alternating sum of consecutive angles between creases must be zero), the Biggest Angle Theorem (all angles in the crease pattern must be less than or equal to 180 degrees), and the Number of Layers Theorem (the number of layers near a flat vertex fold through any point that does not intersect an edge must be even).

Levels Grades 5 through 12, College Geometry, Discrete Math, Combinatorics, Math for Liberal Arts, Introductory Proofs, Mathematical Modeling

Topics Problem Solving, Algebra, Geometry

Goals

- Participants will practice measuring angles with a protractor.
- Participants will consider how new mathematical ideas are developed.
- Participants will experience an open-ended mathematical investigation.
- Participants will make conjectures.
- Participants will support or discard conjectures using evidence and arguments.
- Participants will use symbols and equations to describe the relationships they discover.

Pre-requisite Knowledge Participants need to know how to use a protractor to measure angles.

Preparation Time 5 to 15 minutes

Activity Time 30 minutes for an abbreviated version with no angles, 50 to 60 minutes for version with angles, longer to tackle open research questions.

Materials and Preparation

- 8 small pieces of scrap paper for each person plus some extras (cut full-size paper into four pieces)
- 1 pencil for each person
- 1 protractor for each person
- 1 pair of scissors for each group of four people
- A chalkboard, whiteboard, easel, or overhead projector with writing implements
- 1 handout about related open questions for each person

Primary Source *Project Origami: Activities for Exploring Mathematics* by Thomas Hull, page 165.

Background

Introduce the activity with the following background information. Allow 10 minutes for the background discussion. Allow 5 minutes for the abbreviated version with no angles.

- Mathematics is a way of thinking about situations systematically to discover underlying patterns and relationships.
- Although many math concepts that students learn in school are hundreds or thousands of years old, people create new mathematical ideas all the time.

Ask participants how they think people go about creating new math ideas. Suppose someone sat down and said, “I would like to invent some new math today.” How could they go about doing that? Here are some steps that people might suggest.

- Choose a topic. A new math idea has to be ABOUT something.
- Investigate the topic. It helps to generate lots of examples of the phenomenon you would like to describe.
- Record data about the examples. Counting, measuring, and classifying provide information about the problem.
- Look for a pattern. Listing the data systematically and comparing examples can help with spotting patterns.
- Think of a conjecture. A conjecture is a guess about what is true based on the patterns that you notice.
- Test your conjecture. Look at more examples to see whether the conjecture seems to be true. Try to find counterexamples that break the conjecture. Try to find more examples that fit the pattern. Sometimes a conjecture turns out to be incorrect, but finding that out can lead to new ideas.
- Prove that the conjecture is true. Look for reasons why the conjecture cannot be wrong. Try to argue that the conjecture must always be true no matter what.

Mathematical origami is one fairly new area of research. In this activity, we will investigate a specific problem in mathematical origami relating to flat one-vertex folds. We might re-discover ideas that people have noticed in recent years, or we might think of new ideas. Show participants an example of a flat one-vertex fold and the crease pattern that emerges when it is unfolded. Point out the following things about the fold and its crease pattern.

- The original folded paper must be able to lie flat.
- The creases should be pressed down with a finger or fingernail so they are sharp and easy to see.
- All of the creases on the paper must actually be used. (It is not fair to make some creases and then not fold along them in the final configuration.)
- The crease pattern must have only one place where folds cross each other. This location is called a “vertex”.
- The vertex should lie somewhere in the interior of the paper, not along an edge.

Ask participants to consider what data they could gather about such folds. Some possibilities include:

- Counting total creases.
- Counting “mountain” and “valley” creases separately. (In the opened crease pattern, folds that come up towards you as you look down on the paper are called “mountain” folds. Folds that dip down

away from you are called "valley" folds. It is true that if you turn a piece of paper with a mountain fold upside down, then the fold becomes a valley.)

- Measuring angles between adjacent crease lines. (If necessary, remind participants how to measure angles with a protractor.)
- Counting layers of paper in the original flat fold.
- Considering the way the layers of paper are arranged in the original fold.

Activity Instructions

Participants usually need about 10 minutes to make folds and mark them, 10 minutes to look for patterns and generate conjectures, and 10 more minutes to test and attempt to prove their conjectures. This time can be shortened to 20 minutes in an abbreviated session where participants are not encouraged to measure angles.

- Work in groups of about four people.
- Each person needs 8 pieces of scrap paper, a pencil, a protractor, and scissors.
- Make at least four different flat one-vertex folds with sharp creases. Try to make your folds different from those of other members of the group.
- Open each flat fold to reveal the crease pattern.
- Label mountain folds with the letter "M" and valley folds with the letter "V".
- Measure and label all the angles around the vertex.
- Compare crease patterns with other group members to look for patterns.
- Inspect the layers in the flat folds.
- Try to think of questions, make observations, or make conjectures that might help with this investigation.
- Questions, observations, and conjectures should be announced to everyone as they emerge and should be recorded on the board.

Helpful Hints

It is helpful to suggest the following steps one at a time if participants need help getting into the activity. Participants might be able to think of these hints themselves if asked to consider what data might be useful to record about each crease pattern.

- Try to create conjectures like "I think that ____________ is impossible", "I think that the crease pattern must always ____________", "If the crease pattern has ____________, then ____________", "I notice that ____________". Conjectures might turn out to be incorrect, but a good mathematical discussion requires questions and conjectures that people can consider. Some participants are more comfortable phrasing their conjectures as a question at first.
- How many creases can meet at a vertex in a flat fold?
- Count up the mountains and valleys on each crease pattern and write the totals on each sample.

- It sometimes helps to use scissors to cut off the vertex to investigate the way the layers are arranged near the vertex.
- Before leading this activity, it is extremely helpful to read the additional background information, proofs, and discussion in *Project Origami: Activities for Exploring Mathematics* by Thomas Hull on page 165.
- When students make conjectures encourage them to phrase them precisely and to use equations and variables when appropriate.

Conclusion

Allow about 5 or 10 minutes for the Conclusion.

- Ask participants to discard paper and put away materials.
- Summarize the findings of the group.
- Review the trail of ideas that emerged during the activity. Mention questions, conjectures, examples, counter-examples, and arguments that emerged. Explain that this process is typical of the way that mathematical ideas are discovered.
- If the students are curious about whether their conjectures have been considered before, you may wish to mention related theorems that have been proven or open questions that remain.

Indiana Mathematics Standards

Indiana standards related to this activity include:

- Problem Solving 5.7.1, 6.7.1, 7.7.1, 8.7.1: Analyze problems by identifying relationships, telling relevant from irrelevant information, sequencing and prioritizing information, and observing patterns.
- Problem Solving 6.7.2, 7.7.2, 8.7.2: Make and justify mathematical conjectures based on a general description of a mathematical question or problem.
- Mathematical Reasoning and Problem Solving A1.9.8, A2.10.6, IM1.7.7, IM2.7.9, IM3.7.5: Use counterexamples to show that statements are false, recognizing that a single counterexample is sufficient to prove a general statement false.
- Problem Solving 7.7.5, 8.7.5: Make and test conjectures by using inductive reasoning.
- Mathematical Reasoning and Problem Solving G.8.3, IM2.7.2: Make conjectures about geometric ideas. Distinguish between information that supports a conjecture and the proof of a conjecture.
- Mathematical Reasoning and Problem Solving IM3.7.8: Construct logical arguments, judge their validity, and give counterexamples to disprove statements.
- Problem Solving 5.7.4, 6.7.5, 7.7.6, 8.7.6: Express solutions clearly and logically by using the appropriate mathematical terms and notation. Support solutions with evidence in both verbal and symbolic work.
- Algebra and Functions 5.3.1: Use letters, boxes, or other symbols to represent any number in simple expressions, equations, or inequalities (i.e., demonstrate an understanding of and the use of the concept of a variable).

- Geometry 5.4.1: Measure, identify, and draw right angles, acute angles, obtuse angles, and straight angles using appropriate mathematical tools and technology. (This activity involves practice with measuring angles.)

References

Hull, Thomas. *Project Origami: Activities for Exploring Mathematics.* A. K. Peters; 2006.

Hull, Thomas. "The Combinatorics of Flat Folds: a Survey." *The Proceedings of the Third International Meeting of Origami: Science, Mathematics, and Education.* A. K. Peters; 2002.