Smarties® Sandwiches

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Introduction

The problem below is quoted from an email from Shoshana Sloman, sent on October 26, 2020.

> It has been observed that packs of Smarties® candies can always be evenly divided into five "sandwiches", where a sandwich consists of two same-colored Smarties® with one different-colored candy for the filling. When I say "always", I mean that people who have been playing this "game" for many years have never encountered a roll which couldn't be so divided.
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> I did some research and discovered that there are six possible colors/flavors of Smarties® (orange, green, red, purple, white, and yellow), and that there are 15 per roll. Assuming they were evenly distributed, it makes sense that it would always be possible to create five sandwiches.
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> But further research revealed that the six different colors are mixed up in one huge vat before being randomly placed into individual rolls. So practically speaking they end up fairly evenly distributed, but it's theoretically possible to end up with combinations that could NOT be divided into sandwiches, such as 15 of one color.
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> My thought was that the chances of ending up with an unsandwichable roll would be vanishingly small, and that is why people never observe it. In order to figure this out, I wanted to know the number of sandwichable versus unsandwichable combinations.
>
> Can you tell me how this problem should be approached?

Terminology and Examples

As mentioned in the e-mail above, a Smarties® roll comprises 15 candies randomly selected from the following six colors: red (R), orange (O), yellow (Y), green (G), purple (P), and white (W). Incidentally, the red tablet appears pink and is sometimes described as such; however, we will refer to it as red (R) to avoid confusion with purple (P). A roll can be classified as *sandwichable* or *unsandwichable*, depending on whether it can form five sandwiches. A *sandwich* is a group of three candies consisting of a pair of candies of the same color called a *bun* and one *filling* of a differing color from the color of the bun.

A roll with six green, five yellow, one red, and three purple Smarties® is an example of a sandwichable roll. We could arrange the candies into five triplets, each a sandwich. For example, the arrangement GRG, YPY, GYG, PYP, GYG shows that this roll is sandwichable. A roll with three yellow candies and the rest orange cannot be made into five sandwiches, as there are not five non-orange candies to be the fillings. This is an example of an unsandwichable Smarties® roll. As we will show later (Corollary 1), having more than ten candies of a single color is the necessary and sufficient condition for unsandwichability. We will use this condition to count unsandwichable rolls in the following sections.

Ordered Versus Unordered Rolls

There are two ways to count the total number of possible Smarties® rolls. At first, it might seem reasonable to count rolls without regard to order since we disassemble the rolls to form sandwiches anyway. In other words, we might count PPPPPPPPPPPPPPW and WPPPPPPPPPPPPPP as the same case given their same color profile.

However, it turns out that this approach is incorrect. We actually need to count ordered rolls to find the correct probabilities. This is because while all ordered rolls are equally likely to occur, not every color profile is equally likely. To make this clear, suppose there were only two colors, Green and Purple, and that there were only 4 candies in the roll. There would then be $2^4 = 16$ different possible ordered rolls, but there are 5 possible rolls if we count each color profile once. One of the ordered rolls has 4 Green tablets (GGGG), 4 of them have 3 Green and 1 Purple (GGGP, GGPG, GPGG, PGGG), 6 of them have 2 Green and 2 Purple (GGPP, GPGP, GPPG, PGGP, PGPG, PPGG), 4 of them have 1 Green and 3 Purple (GPPP, PGPP, PPGP, PPPG), and one of them has 4 Purple (PPPP). Because color profiles are not equally likely, we must count ordered Smarties® rolls to obtain the correct probability that a given color combination is manufactured.

Counting Unsandwichable Rolls

We will prove below that, if a Smarties® roll contains 15 candies in at most 6 distinct colors, it is unsandwichable if and only if there are more than 10 candies of any one color. To count the number of rolls that have k candies of one color, we can first count the number of ways to place those k candies into the ordered slots of the roll, then multiply by 6 for the color options for the dominant color, then count the ways we can fill the remaining $15-k$ slots in the roll. This gives us the following expression for each case:

$$\binom{15}{k} \cdot 6 \cdot 5^{15-k}$$

This approach will work as long as $k \geq 8$. Below that value, this expression would over count cases where there are other colors which also have k candies. However, because we are only counting cases with $k \geq 11$, this approach poses no such difficulties.

We will work out each case separately and then add them to find the total number of ordered rolls which are unsandwichable.

Cases with 15 candies of one color

It is easy to see that there are 6 rolls made up of a single color, even without using the expression above.

Cases with 14 candies of one color

$$\binom{15}{14} \cdot 6 \cdot 5^1 = 450$$

Cases with 13 candies of one color

$$\binom{15}{13} \cdot 6 \cdot 5^2 = 15,750$$

Cases with 12 candies of one color

$$\binom{15}{12} \cdot 6 \cdot 5^3 = 341,250$$

Cases with 11 candies of one color

$$\binom{15}{11} \cdot 6 \cdot 5^4 = 5,118,750$$

Overall Probability

As we will show in the next section, every Smarties® roll with at most 10 candies of any one color is sandwichable. Therefore, the total number of unsandwichable rolls can be calculated by summing the above. The result is $6+450+15,750+341,250+5,118,750 = 5,476,206$.

The total number of all possible ordered rolls is $6^{15} = 470,184,984,576$.

Dividing the unsandwichable rolls by the total possible rolls, we find that the probability of finding an unsandwichable roll is about 0.0000116, or .00116%. The Smarties® company sells around 2 billion rolls a year[1], so they should produce about 23,000 unsandwichable rolls every year.

Sandwichability Conditions

Instead of discussing only the case of a roll of Smarties® with 15 candies and 6 colors, in this section we characterize sandwichability for rolls of various lengths and numbers of colors.

Theorem 1 (General Sandwichability Theorem) *Given a roll of Smarties® of length $3n$ for some $n \in \mathbb{N}$, the roll is sandwichable if and only if the following two conditions are fulfilled:*

[1] https://money.cnn.com/2015/10/04/investing/smarties-candy-company-millennial-women/

1. *There are no more than $2n$ Smarties® of any given color.*

2. *There are no more than n colors containing an odd number of Smarties® .*

Proof

Showing that the roll is unsandwichable if the two conditions above are not met is straightforward. Imagine beginning with zero of each color and building the Smarties® roll sandwich by sandwich. Each of the n sandwiches can use at most two of a single color, so we can have at most $2n$ of any given color. Adding a bun to a color cannot change its parity. Therefore, a sandwich can only change the parity of one color with its filling, and since we started with all even colors, we will end up with at most n odd colors. Thus, any roll that does not meet the two conditions above will be unsandwichable.

Next, we will prove that these two conditions are sufficient to guarantee sandwichability. First, we need to show that we can produce n buns. We take away one candy from each odd color. The conditions above guarantee that we will be left with at least $2n$ candies comprised of even colors, so these candies can be formed into n buns.

Next, we will take n arbitrary buns and assign fillings to them arbitrarily. If all of the sandwiches that we have made are legal, we are done. Suppose, then, that there is some illegal sandwich whose filling and bun are the same color, say, orange. Not all of the buns are orange, because if they were and the filling of the illegal sandwich was also orange, there would be at least $2n+1$ orange candies. This contradicts our assumptions.

So let us take some sandwich whose bun is some color besides orange, say, red. There are two cases. In Case 1, the filling of the red bun is some color besides orange, say, green. Then we will put the green filling in the orange bun and the orange filling in the red bun. In Case 2, the filling of the red bun is orange. Then we have four orange candies and two red candies in the two sandwiches, so we will make these into two sandwiches, each with an orange bun and a red filling.

Using the procedure above, we can always reduce the number of illegal sandwiches by one. We will do this for each illegal sandwich until all of the sandwiches are legal and the roll is sandwiched. Thus, any roll of length $3n$ that meets these two conditions is sandwichable, *Οπερ Εδει Δειξαι*

Corollary 1 *If and only if a standard roll of 15 Smarties® has no more than 10 candies of a single color, it is sandwichable.*

Proof: Recall that a standard roll of Smarties® can have up to six colors. There cannot be six odd colors, for then there would be an even number of Smarties® in the roll. The very thing is absurd because 15 is not even. Therefore, there are at most five odd colors. The condition that there are no more than 10 candies of a single color is the other condition of the General Sandwichability Theorem. Therefore, a standard Smarties® roll is unsandwichable if and only if there are more than 10 candies of a single color. *Οπερ Εδει Δειξαι*

Trivia

- Smarties® are known as *Rockets* in Canada because Canada already had a different candy by the same name.

- Classic Smarties® flavors:
  - Red (listed as *pink* on the Smarties® website) = Cherry
  - Orange = Orange
  - Yellow = Pineapple
  - Green = Strawberry
  - Purple = Grape
  - White = Orange-Cream

- Smarties® were invented in 1949 when pellet machines from the war were purchased by the Ce De Candy company and used to make candy.

- Smarties® Webpage
  https://www.smarties.com/

- Video of how Smarties® are made
  Inside a Smarties® Factory:
  https://www.youtube.com/watch?v=PhDux1hdLOY